Saadati *Journal of Inequalities and Applications* (2016) 2016:205
DOI 10.1186/s13660-016-1148-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Existence and uniqueness of solutions for a class of integral equations by common fixed point theorems in IFMT-spaces

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Abstract

In this paper, our aim is to address the existence and uniqueness of solutions for a class of integral equations in IFMT-space. Therefore, we introduce the concept of IFMT-spaces and prove a common fixed point theorem in a complete IFMT-space; next we study an application.

MSC: 54E40; 54E35; 54H25

Keywords: integral equations; nonlinear IF contractive mapping; complete IFMT-space; fixed point theorem

1 Introduction and preliminaries

First of all, we would like to introduce the concept of IFMT-space, which is a non-trivial generalization of IFM-space introduced by Park [1] and Saadati and Park [2] and Saadati *et al.* [3]; also we use results from [4–8].

We say the pair $(L^*, \leq_{L^*})$ is a complete lattice whenever L^* is a non-empty set and we have the operation $\leq_{L^*}$ defined by

$$L^* = \{(a,b) : (a,b) \in [0,1] \times [0,1] \text{ and } a + b \leq 1\},$$

$(a,b) \leq_{L^*} (c,d) \Longleftrightarrow a \leq c$, and $b \geq d$, for each $(a,b),(c,d) \in L^*$.

Definition 1.1 ([9]) An IF set $\mathcal{F}_{\alpha,\beta}$ in a universe $\mathcal{U}$ is an object $\mathcal{F}_{\alpha,\beta} = \{(\alpha_{\mathcal{F}}(u), \beta_{\mathcal{F}}(u)) | u \in \mathcal{U}\}$, in which, for all $u \in \mathcal{U}$, $\alpha_{\mathcal{F}}(u) \in [0,1]$, and $\beta_{\mathcal{F}}(u) \in [0,1]$ are said the membership degree and the non-membership degree, respectively, of u in $\mathcal{F}_{\alpha,\beta}$, and furthermore they satisfy $\alpha_{\mathcal{F}}(u) + \beta_{\mathcal{F}}(u) \leq 1$.

We consider $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$ as its units.

Definition 1.2 ([4]) The mapping $\mathcal{T} : L^* \times L^* \longrightarrow L^*$ satisfying the following conditions:

$(\forall a \in L^*)\ (\mathcal{T}(a, 1_{L^*}) = a)$,
$(\forall (a,b) \in L^* \times L^*)\ (\mathcal{T}(a,b) = \mathcal{T}(b,a))$,
$(\forall (a,b,c) \in L^* \times L^* \times L^*)\ (\mathcal{T}(a, \mathcal{T}(b,c)) = \mathcal{T}(\mathcal{T}(a,b),c))$,
$(\forall (a,a',b,b') \in L^* \times L^* \times L^* \times L^*)\ (a \leq_{L^*} a' \text{ and } b \leq_{L^*} b' \Longrightarrow \mathcal{T}(a,b) \leq_{L^*} \mathcal{T}(a',b'))$.

is said to be a triangular norm (t-norm) on L^*.

$\mathcal{T}$ is said to be a *continuous* t-norm if the triple $(L^*, \leq_{L^*}, \mathcal{T})$ is an Abelian topological monoid with unit 1_{L^*}.

Definition 1.3 ([4]) $\mathcal{T}$ on L^* is called continuous *t-representable* if and only if there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on $[0,1]$ such that, for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$,

$$\mathcal{T}(a,b) = (a_1 * b_1, a_2 \diamond b_2).$$

For example, $\mathcal{T}(a,b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^* is a continuous t-representable.

Definition 1.4 The decreasing mapping $\mathcal{N} : L^* \longrightarrow L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$ is said a negator on L^*. We say $\mathcal{N}$ is an involutive negator if $\mathcal{N}(\mathcal{N}(a)) = a$, for all $a \in L^*$. The decreasing mapping $N : [0,1] \longrightarrow [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$ is said to be a negator on $[0,1]$. The standard negator on $[0,1]$ is defined, for all $a \in [0,1]$, by $N_s(a) = 1 - a$, denoted by N_s. We show $(N_s(a), a) = \mathcal{N}_s(a)$.

Definition 1.5 If for given $\alpha \in (0,1)$ there is $\beta \in (0,1)$ such that

$$\mathcal{T}^m\big(\mathcal{N}_s(\beta), \ldots, \mathcal{N}_s(\beta)\big) >_{L^*} \mathcal{N}_s(\alpha), \quad m \in \mathbf{N},$$

then $\mathcal{T}$ is a *H-type* t-norm.

A typical example of such t-norms is

$$\wedge(a,b) = \big(\mathrm{Min}(a_1, b_1), \mathrm{Max}(a_2, b_2)\big),$$

for every $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^*.

Definition 1.6 The tuble $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is said to be an *IFMT-space* if X is an (non-empty) set, $\mathcal{T}$ is a continuous t-representable, and $\mathcal{M}_{M,N}$ is a mapping $X^2 \times [0, +\infty) \to L^*$ (in which M, N are fuzzy sets from $X^2 \times [0, +\infty)$ to $[0,1]$ such that $M(x,y,t) + N(x,y,t) \leq 1$ for all $x, y \in X$ and $t > 0$) satisfying the following conditions for every $x, y, z \in X$ and $t, s > 0$:
(a) $\mathcal{M}_{M,N}(x,y,t) >_L 0_{L^*}$;
(b) $\mathcal{M}_{M,N}(x,y,t) = \mathcal{M}_{M,N}(y,x,t) = 1_{L^*}$ iff $x = y$;
(c) $\mathcal{M}_{M,N}(x,y,t) = \mathcal{M}_{M,N}(y,x,t)$ for each $x, y \in X$;
(d) $\mathcal{M}_{M,N}(x,y,K(t+s)) \geq_{L^*} \mathcal{T}(\mathcal{M}_{M,N}(x,z,t), \mathcal{M}_{M,N}(z,y,s))$ for some constant $K \geq 1$;
(e) $\mathcal{M}_{M,N}(x,y,\cdot) : [0,\infty) \longrightarrow L^*$ is continuous.

Also $\mathcal{M}_{M,N}$ is said an *IFMT*. Note that for an IFMT-space

$$\mathcal{M}_{M,N}(x,y,t) = \big(M(x,y,t), N(x,y,t)\big).$$

$(X, \mathcal{M}_{M,N}, \mathcal{T})$ is called a *Menger* IFMT-space if

$$\lim_{t\to\infty} \mathcal{M}_{M,N}(x,y,t) = \lim_{t\to\infty} \mathcal{M}_{M,N}(y,x,t) = 1_{L^*}.$$

Remark 1.7 The space of all real functions $\alpha(x)$, $x \in [0,1]$ such that $\int_0^1 |\alpha(x)|^q\,dx < \infty$, denoted by L_q ($0 < q < 1$), is a metric type space. Consider

$$d(\alpha,\beta) = \left(\int_0^1 \left|\alpha(x) - \beta(x)\right|^q\,dx\right)^{\frac{1}{q}},$$

for each $\alpha, \beta \in L_q$. Then d is a metric type space with $K = 2^{\frac{1}{q}}$.

Example 1.8 We consider the set of Lebesgue measurable functions on $[0,1]$ such that $\int_0^1 |\alpha(x)|^q\,dx < \infty$, where $q > 0$ is a real number denoted by $\mathfrak{M}$. Consider

$$\mathcal{M}_{M,N}(x,y,t) = \begin{cases} 0_{L^*} & \text{if } t \leq 0, \\ \left(\frac{t}{t+(\int_0^1 |\alpha(x)-\beta(x)|^q\,dx)^{\frac{1}{q}}}, \frac{(\int_0^1 |\alpha(x)-\beta(x)|^q\,dx)^{\frac{1}{q}}}{t+(\int_0^1 |\alpha(x)-\beta(x)|^q\,dx)^{\frac{1}{q}}}\right) & \text{if } t > 0. \end{cases}$$

So from Remark 1.7, we have $(M, \mathcal{M}_{M,N}, \wedge)$ is IFMT-space with $K = 2^{\frac{1}{q}}$.

Definition 1.9 Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a Menger IFMT-space.

(1) A sequence $\{x_n\}_n$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda \in 0$, there exists a positive integer N such that $\mathcal{M}_{M,N}(x_n, x, \epsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}_n$ in X is called a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda L^* - \{0_{L^*}\}$, there exists a positive integer N such that $\mathcal{M}_{M,N}(x_n, x_m, \epsilon) >_L \mathcal{N}(\lambda)$ whenever $n, m \geq N$.

(3) A Menger IFMT-space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Remark 1.10 Khamsi and Kreinovich [10] proved, if $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is a IFMT-space and $\{u_n\}$ and $\{v_n\}$ are sequences such that $u_n \to u$ and $v_n \to v$, then

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(u_n, v_n, t) = \mathcal{M}_{M,N}(u, v, t).$$

Remark 1.11 Let for each $\sigma \in L^* - \{0_{L^*}, 1_{L^*}\}$ there exists a $\varsigma \in L^* - \{0_{L^*}, 1_{L^*}\}$ (which does not depend on n) with

$$\mathcal{T}^{n-1}\big(\mathcal{N}(\varsigma), \ldots, \mathcal{N}(\varsigma)\big) >_L \mathcal{N}(\sigma) \quad \text{for each } n \in \{1, 2, \ldots\}. \tag{1}$$

Lemma 1.12 ([11]) *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a Menger IFMT-space. If we define $E_{\varsigma,\mathcal{M}_{M,N}} : X^2 \longrightarrow \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\varsigma,\mathcal{M}_{M,N}}(x,y) = \inf\big\{t > 0 : \mathcal{M}_{M,N}(x,y,t) >_L \mathcal{N}(\varsigma)\big\}$$

for each $\varsigma \in L^* - \{0_{L^*}, 1_{L^*}\}$ and $x, y \in X$, then we have the following:

(1) *For any $\sigma \in L^* - \{0_{L^*}, 1_{L^*}\}$, there exists a $\varsigma \in L^* - \{0_{L^*}, 1_{L^*}\}$ such that*

$$E_{\mu,\mathcal{M}_{M,N}}(x_1, x_k) \leq K E_{\varsigma,\mathcal{M}_{M,N}}(x_1, x_2) + K^2 E_{\varsigma,\mathcal{M}_{M,N}}(x_2, x_3) + \cdots + K^{n-1} E_{\varsigma,\mathcal{M}_{M,N}}(x_{k-1}, x_k)$$

for any $x_1, \ldots, x_k \in X$.

(2) *For each sequence* $\{x_n\}$ *in* X, *we have* $\mathcal{M}_{M,N}(x_n, x, t) \longrightarrow 1_{L^*}$ *if and only if* $E_{\varsigma,\mathcal{M}_{M,N}}(x_n, x) \to 0$. *Also the sequence* $\{x_n\}$ *is Cauchy w.r.t.* $\mathcal{M}_{M,N}$ *if and only if it is Cauchy with* $E_{\varsigma,\mathcal{M}_{M,N}}$.

2 Common fixed point theorems

In this section we study some common fixed point theorems in Menger IFMT-spaces, ones can find similar results in others spaces at [12–19].

Definition 2.1 Let f and g be mappings from a Menger IFMT-space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ into itself. The mappings f and g are called weakly commuting if

$$\mathcal{M}_{M,N}(fgx, gfx, t) \geq_L \mathcal{M}_{M,N}(fx, gx, t)$$

for each x in X and $t > 0$.

Now we assume that Φ is the set of all functions

$$\phi : [0, \infty) \longrightarrow [0, \infty)$$

which satisfy $\lim_{n\to\infty} \phi^n(t) = 0$ for $t > 0$ and are onto and strictly increasing. Also, we denote by $\phi^n(t)$ the nth iterative function of $\phi(t)$.

Remark 2.2 Note that $\phi \in \Phi$ implies that $\phi(t) < t$ for $t > 0$. Consider $t_0 > 0$ with $t_0 \leq \phi(t_0)$. Since ϕ is a nondecreasing function we get $t_0 \leq \phi^n(t_0)$ for every $n \in \{1, 2, \ldots\}$, which is a contradiction. Also $\phi(0) = 0$.

Lemma 2.3 ([11]) *If a Menger IFMT-space* $(X, \mathcal{M}_{M,N}, \mathcal{T})$ *obeys the condition*

$$\mathcal{M}_{M,N}(x, y, t) = C, \quad \textit{for all } t > 0,$$

then we get $C = 1_{L^*}$ *and* $x = y$.

Theorem 2.4 *Consider the complete Menger IFMT-space* $(X, \mathcal{M}_{M,N}, \mathcal{T})$. *Assume that* f *and* g *are weakly commuting self-mappings of* X *such that*:

(a) $f(X) \subseteq g(X)$;

(b) f *or* g *is continuous*;

(c) $\mathcal{M}_{M,N}(fx, fy, \phi(t)) \geq_L \mathcal{M}_{M,N}(gx, gy, t)$ *in which* $\phi \in \Phi$.

(i) *Now let* (1) *hold and let there exist a* $x_0 \in X$ *with*

$$E_{\mathcal{M}_{M,N}}(gx_0, fx_0) = \sup\{E_{\gamma,\mathcal{M}_{M,N}}(gx_0, fx_0) : \gamma \in L^* - \{0_{L^*}, 1_{L^*}\}\} < \infty,$$

therefore f *and* g *have a common fixed point which is unique.*

Proof (i) Select $x_0 \in X$ with $E_{\mathcal{M}_{M,N}}(gx_0, fx_0) < \infty$. Select $x_1 \in X$ with $fx_0 = gx_1$. Now select x_{n+1} such that $fx_n = gx_{n+1}$. Now $\mathcal{M}_{M,N}(fx_n, fx_{n+1}, \phi^{n+1}(t)) \geq_L \mathcal{M}_{M,N}(gx_n, gx_{n+1}, \phi^n(t)) = \mathcal{M}_{M,N}(fx_{n-1}, fx_n, \phi^n(t)) \geq_L \cdots \geq \mathcal{M}_{M,N}(gx_0, gx_1, t)$.

We have for each $\lambda \in L^* - \{0_{L^*}, 1_{L^*}\}$ (see Lemma 1.9 of [11])

$$\begin{aligned} E_{\lambda,\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) &= \inf\bigl\{\phi^{n+1}(t) > 0 : \mathcal{M}_{M,N}\bigl(fx_n, fx_{n+1}, \phi^{n+1}(t)\bigr) >_L \mathcal{N}(\lambda)\bigr\} \\ &\leq \inf\bigl\{\phi^{n+1}(t) > 0 : \mathcal{M}_{M,N}(gx_0, fx_0, t) >_L \mathcal{N}(\lambda)\bigr\} \\ &\leq \phi^{n+1}\bigl(\inf\bigl\{t > 0 : \mathcal{M}_{M,N}(gx_0, fx_0, t) >_L \mathcal{N}(\lambda)\bigr\}\bigr) \\ &= \phi^{n+1}\bigl(E_{\lambda,\mathcal{M}_{M,N}}(gx_0, fx_0)\bigr) \\ &\leq \phi^{n+1}\bigl(E_{\mathcal{M}_{M,N}}(gx_0, fx_0)\bigr). \end{aligned}$$

Thus $E_{\lambda,\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) \leq \phi^{n+1}(E_{\mathcal{M}_{M,N}}(gx_0, fx_0))$ for each $\lambda \in L^* - \{0_{L^*}, 1_{L^*}\}$ and so

$$E_{\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) \leq \phi^{n+1}\bigl(E_{\mathcal{M}_{M,N}}(gx_0, fx_0)\bigr).$$

Let $\epsilon > 0$. Select $n \in \{1, 2, \ldots\}$; therefore $E_{\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) < \frac{\epsilon-\phi(\epsilon)}{K}$. For $\lambda \in L^* - \{0_{L^*}, 1_{L^*}\}$ there exists a $\mu \in L^* - \{0_{L^*}, 1_{L^*}\}$ with

$$\begin{aligned} E_{\lambda,\mathcal{M}_{M,N}}(fx_n, fx_{n+2}) &\leq KE_{\mu,\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) + KE_{\mu,\mathcal{M}_{M,N}}(fx_{n+1}, fx_{n+2}) \\ &\leq KE_{\mu,\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) + \phi\bigl(KE_{\mu,\mathcal{M}_{M,N}}(fx_n, fx_{n+1})\bigr) \\ &\leq KE_{\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) + \phi\bigl(KE_{\mathcal{M}_{M,N}}(fx_n, fx_{n+1})\bigr) \\ &\leq K\frac{\epsilon - \phi(\epsilon)}{K} + \phi\left(K\frac{\epsilon - \phi(\epsilon)}{K}\right) \\ &\leq \epsilon. \end{aligned}$$

We can continue this process for every $\lambda \in L^* - \{0_{L^*}, 1_{L^*}\}$; then

$$E_{\mathcal{M}_{M,N}}(fx_n, fx_{n+2}) \leq \epsilon.$$

For $\lambda \in L^* - \{0_{L^*}, 1_{L^*}\}$ there exists a $\mu \in L^* - \{0_{L^*}, 1_{L^*}\}$ with

$$\begin{aligned} E_{\lambda,\mathcal{M}_{M,N}}(fx_n, x_{n+3}) &\leq KE_{\mu,\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) + KE_{\mu,\mathcal{M}_{M,N}}(fx_{n+1}, fx_{n+3}) \\ &\leq KE_{\mu,\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) + \phi\bigl(KE_{\mu,\mathcal{M}_{M,N}}(fx_n, fx_{n+2})\bigr) \\ &\leq KE_{\mathcal{M}_{M,N}}(fx_n, fx_{n+1}) + \phi\bigl(KE_{\mathcal{M}_{M,N}}(fx_n, fx_{n+2})\bigr) \\ &\leq \epsilon - \phi(\epsilon) + \phi(\epsilon) = \epsilon, \end{aligned}$$

from $\mathcal{M}_{M,N}(fx_{n+1}, fx_{n+3}, \phi(t)) \geq_L \mathcal{M}_{M,N}(gx_{n+1}, gx_{n+3}, t) = \mathcal{M}_{M,N}(fx_n, fx_{n+2}, t)$ we have $E_{\lambda,\mathcal{M}_{M,N}}(fx_{n+1}, fx_{n+3}) \leq \phi(E_{\mu,\mathcal{M}_{M,N}}(fx_n, fx_{n+2}))$, which implies that

$$E_{\mathcal{M}_{M,N}}(fx_n, fx_{n+3}) \leq \epsilon.$$

By using induction

$$E_{\mathcal{M}_{M,N}}(fx_n, fx_{n+k}) \leq \epsilon \quad \text{for } k \in \{1, 2, \ldots\},$$

and we conclude that $\{fx_n\}_n$ is a Cauchy sequence and by the completeness of X, $\{fx_n\}_n$ converges to a point named z in X. Also $\{gx_n\}_n$ converges to z. Now we assume that the mapping f is continuous. Then $\lim_n ffx_n = fz$ and $\lim_n fgx_n = fz$. Also, since f and g are weakly commuting,

$$\mathcal{M}_{M,N}(fgx_n, gfx_n, t) \geq_L \mathcal{M}_{M,N}(fx_n, gx_n, t).$$

Take $n \to \infty$ in the above inequality and we get $\lim_n gfx_n = fz$, by the continuity of $\mathcal{M}$. Now, we show that $z = fz$. Assume that $z \neq fz$. From (c) for each $t > 0$ we have

$$\mathcal{M}_{M,N}\big(fx_n, ffx_n, \phi^{k+1}(t)\big) \geq_L \mathcal{M}_{M,N}\big(gx_n, gfx_n, \phi^k(t)\big), \quad k \in \mathbb{N}.$$

Suppose that $n \to \infty$ in the above inequality; we get

$$\mathcal{M}_{M,N}\big(z, fz, \phi^{k+1}(t)\big) \geq_L \mathcal{M}_{M,N}\big(z, fz, \phi^k(t)\big).$$

Furthermore we have

$$\mathcal{M}_{M,N}\big(z, fz, \phi^k(t)\big) \geq_L \mathcal{M}_{M,N}\big(z, fz, \phi^{k-1}(t)\big)$$

and

$$\mathcal{M}_{M,N}\big(z, fz, \phi(t)\big) \geq_L \mathcal{M}_{M,N}(z, fz, t).$$

Also

$$\mathcal{M}_{M,N}\big(z, fz, \phi^{k+1}(t)\big) \geq_L \mathcal{M}_{M,N}(z, fz, t).$$

Next, we have (see Remark 2.2)

$$\mathcal{M}_{M,N}\big(z, fz, \phi^{k+1}(t)\big) \leq_L \mathcal{M}_{M,N}(z, fz, t).$$

Then $\mathcal{M}_{M,N}(z, fz, t) = C$ and from Lemma 2.3, we conclude that $z = fz$. By assumption we have $f(X) \subseteq g(X)$; then there exists a z_1 in X such that $z = fz = gz_1$. Now,

$$\mathcal{M}_{M,N}(ffx_n, fz_1, t) \geq_L \mathcal{M}_{M,N}\big(gfx_n, gz_1, \phi^{-1}(t)\big).$$

Take $n \to \infty$; we get

$$\mathcal{M}_{M,N}(fz, fz_1, t) \geq_L \mathcal{M}_{M,N}\big(fz, gz_1, \phi^{-1}(t)\big) = 1_{L^*},$$

then $fz = fz_1$, *i.e.*, $z = fz = fz_1 = gz_1$. Also for each $t > 0$ we get

$$\mathcal{M}_{M,N}(fz, gz, t) = \mathcal{M}_{M,N}(fgz_1, gfz_1, t) \geq_L \mathcal{M}_{M,N}(fz_1, gz_1, t) = \varepsilon_0(t)$$

since f and g are weakly commuting, from which we can conclude that $fz = gz$. This implies that z is a common fixed point of f and g.

Now we prove the uniqueness. Assume that $z' \neq z$ is another common fixed point of f and g. Now, for each $t > 0$ and $n \in \mathbb{N}$, we have

$$\mathcal{M}_{M,N}\bigl(z, z', \phi^{n+1}(t)\bigr) = \mathcal{M}_{M,N}\bigl(fz, fz', \phi^{n+1}(t)\bigr) \geq_L F_{gz,gz'}\bigl(\phi^n(t)\bigr) = F_{z,z'}\bigl(\phi^n(t)\bigr).$$

Also of course we have

$$\mathcal{M}_{M,N}\bigl(z, z', \phi^n(t)\bigr) \geq_L \mathcal{M}_{M,N}\bigl(z, z', \phi^{n-1}(t)\bigr)$$

and

$$\mathcal{M}_{M,N}\bigl(z, z', \phi^n(t)\bigr) \geq_L \mathcal{M}_{M,N}\bigl(z, z', t\bigr).$$

As a result

$$\mathcal{M}_{M,N}\bigl(z, z', \phi^{n+1}(t)\bigr) \geq_L \mathcal{M}_{M,N}\bigl(z, z', t\bigr).$$

On the other hand we have

$$\mathcal{M}_{M,N}\bigl(z, z', t\bigr) \leq_L \mathcal{M}_{M,N}\bigl(z, z', \phi^{n+1}(t)\bigr).$$

Then $\mathcal{M}_{M,N}(z, z', t) = C$, see Lemma 2.3, implies that $z = z'$, which is contradiction. Then z is the unique common fixed point of f and g. □

3 The existence and uniqueness of solutions for a class of integral equations

Assume that $X = C([1,3], (-\infty, 2.1443888))$ and

$$\mathcal{M}_{M,N}(x, y, t) = \begin{cases} 0 & \text{if } t \leq 0, \\ (\inf_{\ell\in[1,3]} \frac{t}{t+(x(\ell)-y(\ell))^2}, \sup_{\ell\in[1,3]} \frac{(x(\ell)-y(\ell))^2}{t+(x(\ell)-y(\ell))^2}) & \text{if } t > 0, \end{cases}$$

for $x, y \in X$, then $(M, \mathcal{M}_{M,N}, \wedge)$ is a complete IFTM-space with $K = 2$.

We consider the mapping $T : X \to X$ by

$$T\bigl(x(\ell)\bigr) = 4 + \int_1^\ell \bigl(x(u) - u^2\bigr)e^{1-u}\, du.$$

Put $g(x) = T(x)$ and $f(x) = T^2(x)$. Since $fg = gf$, f and g are (weakly) commuting. Now, for $x, y \in X$ and $t > 0$,

$$\begin{aligned} &\mathcal{M}_{M,N}(fx, fy, t) \\ &\quad = \mathcal{M}_{M,N}\bigl(T\bigl(Tx(\ell)\bigr), T\bigl(Ty(\ell)\bigr), t\bigr) \\ &\quad = \left(\inf_{\ell\in[1,3]} \frac{t}{t + |\int_1^\ell (Tx(u) - Ty(u))e^{1-u}\, du|^2}, \sup_{\ell\in[1,3]} \frac{|\int_1^\ell (Tx(u) - Ty(u))e^{1-u}\, du|^2}{t + |\int_1^\ell (Tx(u) - Ty(u))e^{1-u}\, du|^2}\right) \\ &\quad \geq \left(\frac{t}{t + \frac{1}{e^4}|\int_1^3 (Tx(u) - Ty(u))\, du|^2}, \frac{\frac{1}{e^4}|\int_1^3 (Tx(u) - Ty(u))\, du|^2}{t + \frac{1}{e^4}|\int_1^3 (Tx(u) - Ty(u))\, du|^2}\right) \\ &\quad = \mathcal{M}_{M,N}(gx, gy, t), \end{aligned}$$

then

$$\mathcal{M}_{M,N}\left(fx,fy,\left(\frac{t}{e^4}\right)\right) \geq_L \mathcal{M}_{M,N}(gx,gy,t).$$

Thus all conditions of Theorem 2.4 are satisfied for $\phi(t) = \frac{t}{e^4}$ and so f and g have a unique common fixed point, which is the unique solution of the integral equations

$$x(\ell) = 4 + \int_1^\ell \left(x(u) - u^2\right)e^{1-u}\,du$$

and

$$x(\ell) = (1-\ell)^2 e^{1-\ell} + \int_1^\ell \int_1^u \left(x(v) - v^2\right)e^{2-(u+v)}\,dv\,du.$$

Competing interests
The author declares to have no competing interests.

Author's contributions
Only the author contributed in writing this paper.

Acknowledgements
The author is grateful to the reviewers for their valuable comments and suggestions.

Received: 24 April 2016 Accepted: 13 August 2016 Published online: 24 August 2016

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